stresses that often occur for small values of tb/r for impulse-type heat fluxes.

The corresponding formula for $a = 0$ has the basic solution[2] $\Delta_2(r,t)$ with

$$\Delta_2(r,t) \backsim \frac{1}{k}\left[2\left(\frac{\alpha bt}{r}\right)^{1/2} i \operatorname{erfc}\left(\frac{b-r}{2(\alpha t)^{1/2}}\right) + \frac{\alpha t(b+3r)}{2(br^3)^{1/2}} i^2 \operatorname{erfc}\left(\frac{b-r}{2(\alpha t)^{1/2}}\right)\ldots\right] \quad (24)$$

and $T_2(r,t) = T_1(r,t)$ to this order of accuracy.

It will be observed, comparing Eq. (22) and Eq. (24), that the first terms of this expansion are independent of the inner radius of the cylinder. Physically, this shows that, for the initial temperature changes only, the reflection of the "temperature wave" at the inner boundary may be neglected, and the temperature may be obtained as if the cylinder were solid.

As time goes on, however, this approximation will get worse, but for impulse-type heat fluxes it should be sufficiently accurate to predict maximum heating rates. The first form for T [Eq. (14)] is useful when $Q'(t - \tau)$ is an impulse-type function, so that $Q(t - \tau)$ is a step function.

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An Alternate Interpretation of Newton's Second Law

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Introduction

THE derivation of the momentum equation for variable mass systems has been reopened recently for discussion. In these various discussions[1] two assumptions usually are made: 1) the classical momentum equations do not apply to systems of variable mass; and 2) the derivation of the rocket equation requires two separate control volumes containing a total constant mass.

In this paper it will be shown that, although assumption 2 is a result of assumption 1, assumption 2 denies assumption 1. It will be shown that the classical momentum equation for a particle is given in an incomplete form and that Newton's equation (in the classical expression) does lead to the rocket equation.

Assumption 1 thus will be shown to be unnecessary, and assumption 2 will be replaced by a calculus operation. A general definition of momentum, valid for Lebesgue measurable mass,[2] will be developed. The new definition of momentum will be shown to be valid for point masses, summations of mass points, piecewise continuous masses, continuous masses, and for both time variable and time invariant masses.

Standard Derivation of "Rocket" Equation

It is customary to define the law of linear momentum in terms of the mass m_i, velocity $\mathbf{v}_i$, and the net force $\mathbf{F}_i$, acting on the ith particle of a system of particles:[3]

$$\mathbf{F}_i = m_i(d\mathbf{v}_i/dt) = (d/dt)(m_i\mathbf{v}_i) \quad (1)$$

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For a collection of n particles, Newton's second law becomes

$$\Sigma_i \mathbf{F}_i = \Sigma_i(d/dt)(m_i \mathbf{v}_i)$$

If the number of particles is time constant, the summation and the differentiation can be interchanged

$$\Sigma_i \mathbf{F}_i = (d/dt)\Sigma_i m_i\mathbf{v}_i = d\mathbf{G}/dt \quad (2)$$

in which $\mathbf{G}$ is the total linear momentum.

Obviously, the derivation of Eq. (2) requires $\mathbf{G}$ to be the momentum of a system of constant mass. It is possible, however, to use Eq. (2) on variable mass system by the following artifice.

Let there be two volumes Y and Y_E having a common boundary. Let Y be the control volume for which the momentum $\mathbf{G}$ is to be found, and let the momentum of the particles in Y_E be $\mathbf{G}_E$. If it is assumed further that the total mass in the two volumes is a constant, then Eq. (2) applies:

$$\Sigma_i \mathbf{F}_i = (d/dt)(\mathbf{G} + \mathbf{G}_E) = (d\mathbf{G}/dt) + (d\mathbf{G}_E/dt) \quad (3)$$

The second term on the right of Eq. (3) seems to imply that forces acting on Y_E should have an effect on Y. This physically indefensible result can be removed by letting the volume Y_E approach zero. In the limit, $d\mathbf{G}_E/dt$ simply becomes the rate at which momentum crosses the boundary of Y. Equation (3) is the expression for the momentum theorem of variable mass systems.

Equation (3) can be applied to a single particle that enters Y at time T_1 and leaves at time T_2:

1) For time $t < T_1$, Eq. (3) is identically zero.

2) For time $t > T_2 > T_1$, Eq. (3) is identically zero.

3) At times T_1 and T_2, $d\mathbf{G}/dt = md\mathbf{v}/dt$, and $d\mathbf{G}_E/dt$ represents an impulsive change of momentum on the boundary.

If one defines $u_1(t - T_1)$ as a unit step function open on the left and $u_2(t - T_2)$ a unit step function open on the right, Eq. (3) becomes for a single particle

$$[u_1(t - T_1) - u_2(t - T_2)]\mathbf{F}_i = [u_1(t - T_1) - u_2(t - T_2)]m_i(d\mathbf{v}_i/dt) + \delta(t - T_1)m_i\mathbf{v}_i - \delta(t - T_2)m_i\mathbf{v}_i \quad (4)$$

in which $\delta(t - T)$ is the Dirac Delta or unit impulse function.[4] It is well known that

$$(d/dt)[u(t - T)] = \delta(t - T)$$

Thus Eq. (4) becomes

$$A_i(T_1,T_2,t)\mathbf{F}_i = (d/dt)[A_i(T_1,T_2,t)m_i\mathbf{v}_i] \quad (5)$$

in which A_i, the closed pulse function, is defined as

$$A_i(T_1,T_2,t) = \begin{cases} 0 & t < T_1 \\ 1 & T_2 \geqslant t \geqslant T_1 \\ 0 & t > T_2 \end{cases}$$

Now the function A_i is nonzero only for particles in the control volume Y; thus one can write the equation for a system of particles by summing over all particles in Y and all particles outside of Y. (This is the division into Y and Y_E, except that now Y_E can be any volume sufficiently large to contain all particles that will be in Y_E at any time and any other additional particles. Y_E could, for example, contain all the particles in the universe except for the particles in Y.)

Thus the momentum equation becomes

$$\sum_{i=1}^{\infty} A_i\mathbf{F}_i = \sum_{i=1}^{\infty}\left[\frac{d}{dt}(A_im_i\mathbf{v}_i)\right]$$

Here the interchange of derivative and summation is valid

$$\sum_{i=1}^{\infty} A_i\,\mathbf{F}_i = \frac{d}{dt}\left(\sum_{i=1}^{\infty} A_i\,m_i\,\mathbf{v}_i\right)$$

but $A_i = 1$ for particles inside Y and zero for particles outside, whence one obtains (assuming n particles inside Y)

$$\sum_{i=1}^{n} \mathbf{F}_i = \frac{d}{dt}\left[\sum_{i=1}^{n} (m_i \mathbf{v}_i)\right] = \frac{d\mathbf{G}}{dt} \tag{6}$$

That this is equivalent to Eq. (3) can be seen immediately by noting that

$$\frac{d}{dt}\left[\sum_{i=1}^{n} (m_i \mathbf{v}_i)\right] = \sum_{i=1}^{n} \frac{d}{dt} (m_i \mathbf{v}_i) + \frac{\partial \mathbf{G}}{\partial n}\frac{dn}{dt} \tag{7}$$

in which the first term on the right is the rate of change of momentum within Y and the second term is the rate at which momentum crosses the boundary. Note that the derivatives exist only in the formal or Dirac sense, since **G** is discontinuous. It can be shown, however, that

$$n = \sum_{i=1}^{\infty} A_i$$

and that $\partial\mathbf{G}/\partial n$ represent an average momentum of the particles crossing the boundary at a given instant.

Equation (6) proves that assumption 1 is incorrect and that the classical momentum equation is valid for variable mass systems. It is possible to obtain Eq. (6) directly from Eq. (3) without recourse to the division of space into Y and Y_E simply by proceeding from Eqs. (3–7) and thence to (6). It also is possible to obtain Eq. (6) by writing the momentum summation at time t and $t + \Delta t$ and taking the limit of the difference. This process yields Eq. (7) directly in terms of unit impulses on the boundary.

General Principle

Within a system, a particle located at (x_i, y_j, z_k) has a mass

$$m_{ijk}u(x - x_i)u(y - y_j)u(z - z_k)$$

The unit step functions are all open on the right. Thus the mass within Y becomes

$$m = \sum_{i,j,k} m_{ijk}u(x - x_i)u(y - y_j)u(z - z_k)$$

It is well known that the Stieltjes-Lebesgue integral with respect to a unit step is[2]

$$f(\xi) = \int f(x)\, d[u(x - \xi)]$$

and therefore

$$\mathbf{G} = \Sigma_i m_i \mathbf{v}_i = \int \mathbf{v}\, dm \tag{8}$$

By a simple limit procedure m may be made continuous, from which it can be seen that Eq. (6) applies to continuous mass systems. That this statement is true can be demonstrated easily. Let m be continuous; then $dm/dY = \rho$ is the mass density and $dY = dxdydz$. Select a system of orthogonal coordinates h_1, h_2, h_3 such that $h_3 = p(t)$ represents the outer boundary of Y; $p(t)$ is a function of time, since the control volume Y is defined by the mass, and mass is moving on the boundary. Then

$$= \int_a^b \int_c^d \int_e^f \mathbf{v}\rho\, dxdydz = \int_0^p \int_0^q \int_0^r \mathbf{v}\rho\, J dh_1 dh_2 dh_3$$

a, b, c, d, e, f, and p are functions of time [these are equivalent to the upper summation limit in Eq. (6)], and J is the Jacobian of the transformation. Applying the standard rule for differentiating under the integral sign,[5] one obtains

$$\frac{d\mathbf{G}}{dt} = \int_0^p \int_0^q \int_0^r \frac{\partial}{\partial t} (\rho \mathbf{v} J) dh_1 dh_2 dh_3 + \frac{dp}{dt} R(p) \tag{9}$$

where

$$R(p) = \int_0^q \int_0^r \mathbf{v}\rho\, J dh_1 dh_2$$

at $h_3 = p$. Equation (9) represents the rocket equation, since the first term on the right is the rate of change of momentum in Y, and the second term is the rate at which momentum crosses the boundary.

Conclusions

1) If the momentum of a mass system is represented by the Stieltjes-Lebesgue integral

$$\mathbf{G} = \int \mathbf{v}\, dm \tag{10}$$

in which the integral is taken over all the mass in the system, then Newton's equation

$$\Sigma\mathbf{F} = d\mathbf{G}/dt \tag{11}$$

is valid for all continuous and discontinuous mass system and for time-variable as well as time-fixed masses.

2) The equation

$$\sum_{i=1}^{n} \mathbf{F}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \mathbf{v}_i\right) \tag{12}$$

is valid for time variable mass systems.

3) In Eq. (12)

$$n = \sum_{i=1}^{\infty} A_i(T_1, T_2, t) \tag{13}$$

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Singular Line of the Method of Integral Relations

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IN the method of integral relations, as applied by Bielotserkovskii to the blunt-body problem,[1–3] there occurs a singular line that cuts across the shock layer. The equation of the singular line is obtained by equating the determinant of the system of resulting ordinary differential equations to zero. This equation, however, gives no information about the position of the singular line, and its physical significance has not been explained in the literature.

Bielotserkovskii's method has been criticized often for its reliance on the boundary conditions given on the singular line. The main objection to such a procedure seemed to be the fact that coordinates of the singular line depend not only on the solution but also on the choice of the coordinate system. This note explains this apparent discrepancy by giving the singular line a physical interpretation.

A problem of a blunt body placed at zero incidence in a supersonic flow is considered. Perfect gas is assumed. Without loss of generality, the s,n coordinate system is chosen,

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